ISRAEL JOURNAL OF MATHEMATICS, Vol. 69, No. 2, 1990

SYMMETRIC BLOCK BASES OF SEQUENCES WITH LARGE AVERAGE GROWTH

BY

W. T. GOWERS

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, England

ABSTRACT

We show that if $0<\varepsilon \leqq 1$, $1 \leqq p<2$ and $x_1, \ldots, x_n$ is a sequence of unit vectors in a normed space X such that $\mathbf{E}\,\|\sum_1^n \varepsilon_i x_i\| \geqq n^{1/p}$, then one can find a block basis $y_1, \ldots, y_m$ of $x_1, \ldots, x_n$ which is $(1+\varepsilon)$-symmetric and has cardinality at least $\gamma n^{2/p-1}(\log n)^{-1}$, where γ depends on ε only. Two examples are given which show that this bound is close to being best possible. The first is a sequence $x_1, \ldots, x_n$ satisfying the above conditions with no 2-symmetric block basis of cardinality exceeding $2n^{2/p-1}$. This sequence is not linearly independent. The second example is a sequence which satisfies a lower p-estimate but which has no 2-symmetric block basis of cardinality exceeding $Cn^{2/p-1}(\log n)^{4/3}$, where C is an absolute constant. This applies when $1 \leqq p \leqq 3/2$. Finally, we obtain improvements of the lower bound when the space X containing the sequence satisfies certain type-conditions. These results extend results of Amir and Milman in [1] and [2]. We include an appendix giving a simple counterexample to a question about norm-attaining operators.

§1. Introduction

The aim of this paper, like that of [7], is to show that if a basis satisfies certain conditions, then it has an almost symmetric block basis of reasonably high cardinality. Where possible, examples are also given to show that the results cannot be substantially improved. The starting point for both papers was a pair of papers by Amir and Milman [1] [2]. They considered, amongst other things, bases that are equivalent to the unit vector basis of l_p^n, for $1 \leqq p<\infty$, and bases that satisfy some growth condition. In this paper, we consider normalized bases $x_1, \ldots, x_n$ that satisfy

Received March 24, 1989 and in revised form June 2, 1989

$$\mathbf{E}\left\|\sum_1^n \varepsilon_i x_i\right\| \geqq n^{1/p}$$

when $1 \leqq p < 2$. We show that, given such a basis, one can find an almost symmetric block basis of cardinality $\gamma n^{2/p-1}/\log n$, where $\gamma = \gamma(\varepsilon)$. We show that if we do not require the original basis to be linearly independent, or else if we ask for some rather natural conditions to be satisfied by the block basis, then this result is, barring the $\log n$ factor, best possible. We also show that it is best possible under any circumstances if $1 \leqq p \leqq 3/2$, again up to a power of $\log n$. The example given for this case happens to satisfy a lower p-estimate. Finally, we consider the effect of imposing a type condition on the space containing the basis, obtaining a few results which suggest that the exact value of the type constant is important. In fact, it seems that we are helped as much by good cotype as by good type: we use a fairly standard argument to relate the cotype of a Banach space to its type constant. Combining this with some well known results and the result of [7], we show that if p is close to 2 and the space containing the basis has a small enough p-type constant, then one can obtain an almost symmetric block basis of cardinality significantly larger than $n^{2/p-1}$. In an appendix, we give a simple counterexample to a question about norm-attaining operators.

We recall the following definitions. A sequence $x_1, \ldots, x_n$ of elements of a Banach space is said to be α-symmetric if for any sequence $a_1, \ldots, a_n$ of scalars, any sequence $\varepsilon_1, \ldots, \varepsilon_n$ of signs and any permutation $\pi \in S_n$, we have

$$\left\|\sum_1^n \varepsilon_i a_i x_{\pi(i)}\right\| \leqq \alpha \left\|\sum_1^n a_i x_i\right\| .$$

If for a particular sequence $(a_i)_1^n$ the above inequality holds we shall say that the basis $(x_i)_1^n$ is α-symmetric at $(a_i)_1^n$, or else α-symmetric at $\mathbf{a}$, where $\mathbf{a} = \Sigma_1^n a_i x_i$.

A block sequence (or block basis) of a sequence $x_1, \ldots, x_n$ is a collection of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ where each $\mathbf{u}_j$ is of the form $\Sigma_{i \in A_j} \lambda_i x_i$ and $A_1, \ldots, A_m$ are all disjoint. The p-type constant $T_p(X)$ of a space X is the smallest constant C for which

$$\left(\mathbf{E}\left\|\sum_1^N \varepsilon_i x_i\right\|^2\right)^{1/2} \leqq C\left(\sum_1^N \|x_i\|^p\right)^{1/p}$$

for all sequences $x_1, \ldots, x_N$ of vectors in X. Here $(\varepsilon_i)_1^N$ is again a sequence of signs, and each sequence occurs with probability 2^{-N}. We denote by $T_p(X, k)$

the smallest constant C for which the above inequality is satisfied whenever $N \leqq k$. Note that $T_p(X, k) \leqq T_p(X)$.

We denote by $\alpha_p(X)$ the smallest constant C for which the above inequality holds when $(\varepsilon_i)_1^N$ is replaced by a sequence $(g_i)_1^N$ of independent random variables all distributed as $N(0, 1)$. We define $\alpha_p(X, k)$ in the obvious way. These two constants are known as the Gaussian p-type constant and the Gaussian p-type constant on k vectors respectively.

The q-cotype constant $C_q(X)$ is the smallest constant C for which the following inequality always holds:

$$\left(\mathbf{E}\left\|\sum_1^N \varepsilon_i x_i\right\|^2\right)^{1/2} \geqq C^{-1}\left(\sum_1^N \| x_i \|^q\right)^{1/q}.$$

We define also $C_q(X, k)$ just as in the type case. We denote by $\beta_q(X)$ and $\beta_q(X, k)$ the best constants when Gaussian variables are used.

Given a scalar sequence $(a_i)_1^n$ we shall denote by $(a_i^*)_1^n$ its positive decreasing rearrangement, that is, the decreasing rearrangement of the sequence $(|a_i|)_1^n$.

With a few exceptions, expressions that strictly speaking take integer values are not presented as integers. This is for the sake of tidiness, and because in this context, it is not important. The scalars are assumed throughout to be real, but the results carry over easily to the complex case.

§2. The main theorem

THEOREM 1. *Let $1 \leqq p < 2$, $0 < \varepsilon \leqq 1$, and let $(x_i)_1^n$ be a sequence of unit vectors in a Banach space X. Suppose that $\mathbf{E} \| \sum_1^n \varepsilon_i x_i \| \geqq n^{1/p}$. Then there is a block basis of $(x_i)_1^n$ with blocks of ± 1 coefficients and equal length, which is $(1 + \varepsilon)$-symmetric and has cardinality*

$$m \geqq \gamma(\varepsilon)(\log n)^{-1} n^{2/p - 1}$$

where $\gamma(\varepsilon) = (\varepsilon^3/3{,}000{,}000)(\log(33/\varepsilon))^{-1}$

This bound improves on the previously known bound of $n^{(2-p)^2/3p^3}$ obtained by Amir and Milman in [2], section 5.2. They have also shown that if X has p-type constant C, one can obtain a bound of $\gamma(\varepsilon, p, C) n^{(2-p)/3p(2+3p)}$ (Theorem 2.4 of [1]).

The notation and structure of the proof of Theorem 1 will be the same as those of Theorem 1 of [7]. For the sake of completeness, much will be repeated.

Let Ψ be the group $\{-1, 1\}^m \times S_m$ with multiplication given by

$$((\eta_i)_1^m, \sigma) \circ ((\eta_i')_1^m, \sigma') = ((\eta_i \eta_i')_1^m, \sigma \circ \sigma'),$$

acting on $\mathbf{R}^m$ as follows. If $\mathbf{a} = \Sigma_1^m a_i \mathbf{e}_i \in \mathbf{R}^m$, and $(\eta, \sigma) \in \Psi$, $\eta = (\eta_i)_1^m$, then

$$(\eta, \sigma) : \mathbf{a} \mapsto \psi_{\eta,\sigma}(\mathbf{a}) \equiv \mathbf{a}_{\eta,\sigma} = \sum_1^m \eta_i a_i \mathbf{e}_{\sigma(i)}.$$

Let Ω be the group $\{-1, 1\}^n \times S_n$ acting on X as follows. If $\mathbf{b} \in X$, $\mathbf{b} = \Sigma_1^n b_i x_i$, and $(\varepsilon, \pi) \in \Omega$, then

$$(\varepsilon, \pi) : \mathbf{b} \mapsto \omega_{\varepsilon,\pi}(\mathbf{b}) \equiv \mathbf{b}_{\varepsilon,\pi} = \sum_1^n \varepsilon_i b_i x_{\pi(i)}.$$

We shall sometimes relabel the indices of $(x_i)_1^n$. Let

$$x_{ij} = x_{(i-1)h+j} \qquad (i = 1, \dots, m, j = 1, \dots, h), \qquad \text{where } hm = n,$$

and, similarly, let $\varepsilon_{ij} = \varepsilon_{(i-1)h+j}$ and $\pi_{ij} = \pi((i-1)h + j)$ for $(\varepsilon, \pi) \in \Omega$.

As in [7] we shall regard a block basis of $(x_i)_1^n$ as a random embedding of $\mathbf{R}^m$ into X. Let $\phi : \mathbf{R}^m \to X$ be the embedding defined by

$$\phi : \sum_{i=1}^m a_i \mathbf{e}_i \mapsto \sum_{i=1}^m \sum_{j=1}^h a_i x_{ij}$$

and write $\mathbf{u}_i = \Sigma_{j=1}^h x_{ij}$, for $i = 1, \dots, m$. Then let $\phi_{\varepsilon,\pi} = \omega_{\varepsilon,\pi} \circ \phi$, i.e.

$$\phi_{\varepsilon,\pi} : \sum_{i=1}^m a_i \mathbf{e}_i \mapsto \sum_{i=1}^m \sum_{j=1}^h \varepsilon_{ij} a_i x_{\pi_{ij}}.$$

When $1 \leqq i \leqq m$ we shall write $(\mathbf{u}_i)_{\varepsilon,\pi}$ for $\phi_{\varepsilon,\pi}(\mathbf{e}_i)$. The sequence $((\mathbf{u}_i)_{\varepsilon,\pi})_1^m$ will be our random block basis.

The proof of Theorem 1 is based on the following two lemmas, which resemble assertions made in [7].

Lemma 2. (i) *Let the sequence $x_1, \dots, x_n \in X$ satisfy the conditions of Theorem* 1, *let q be the conjugate index of p and let $\phi_{\varepsilon,\pi} : \mathbf{R}^m \mapsto X$ be as defined earlier. Then for any $\mathbf{a} \in \mathbf{R}^m$, $\mathbf{E}_\Omega \| \phi_{\varepsilon,\pi}(\mathbf{a}) \| \geqq n^{-1/q} \| \mathbf{a} \|_1 h$.*

(ii) *Let ε and m be as in the statement of Theorem* 1, *and let $\delta = \varepsilon/11$. For any $\mathbf{a} \in A$, let $E(\mathbf{a})$ stand for $\mathbf{E}_\Omega \| \phi_{\varepsilon,\pi}(\mathbf{a}) \|$. Then*

$$\mathbf{P}_\Omega \left[\exists (\eta, \sigma) \text{ s.t.} \left| \, \| \phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma}) \| - E(\mathbf{a}) \right| > \frac{\delta}{3} n^{-1/q} \| \mathbf{a} \|_1 h \right] < m^{-\delta^{-1} \log(3\delta^{-1})}.$$

LEMMA 3. *Let* $\delta > 0$, *let* $(\mathbf{R}^m, \|\cdot\|)$ *be a normed space and set* $N = m^{\delta^{-1}\log(3\delta^{-1})}$. *There exist* N *vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_N$ *such that if* $\|\cdot\|$ *is* $(1+\delta)$-*symmetric at* $\mathbf{a}_i$ *for every* i, *then the standard basis of* $\mathbf{R}^m$ *is* $(1+\delta)(1-6\delta)^{-1}$-*symmetric*.

Lemma 3 plays the role of Lemma 2 of [7], but needs a somewhat different proof. The set of vectors $\{(\mathbf{a}_i)_{\eta,\sigma}: 1 \leqq i \leqq N, (\eta,\sigma) \in \Psi\}$ does not necessarily form a net of the unit ball of the space $(\mathbf{R}^m, \|\cdot\|)$. Instead we make slightly more use of convexity. The main steps in the argument are the two parts of Lemma 2. The second part states, roughly, that for any vector $\mathbf{a} \in \mathbf{R}^m$, the deviation of $\phi_{\varepsilon,\pi}(\mathbf{a})$ from its average is, with large probability, small. The first part provides a lower bound for the average itself, so that the deviation is in fact proportionately small. In fact, Theorem 1 is very similar to the case where $p = 1$ and $C = n^{1/p}$ in Theorem 1 of [7].

Let us see why Lemmas 2 and 3 are enough to prove Theorem 1. If they are both true, then we can choose vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$ that satisfy the conclusion of Lemma 3. Having done this, let us consider a single vector $\mathbf{a} = \mathbf{a}_i$. Taking $\delta = \varepsilon/11$, we have $\delta < 1/11$, so certainly $(1+\delta/3)(1-\delta/3)^{-1} \leqq 1+\delta$. Hence, by the two parts of Lemma 2,

$$\mathbf{P}_\Omega\left[\max_{\Psi} \|\phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma})\| \Big/ \min_{\Psi} \|\phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma})\| > 1+\delta\right] < m^{-\delta^{-1}\log(3\delta^{-1})} = N^{-1}.$$

It follows that there exists some $(\varepsilon,\pi) \in \Omega$ such that if we define a norm of $\mathbf{R}^m$ by $\|\mathbf{a}\| \equiv \|\phi_{\varepsilon,\pi}(\mathbf{a})\|$, then this norm is $(1+\delta)$-symmetric at each of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_N$. But then by Lemma 3 this norm is $(1+\delta)(1-6\delta)^{-1}$-symmetric on $\mathbf{R}^m$. But since $\delta = \varepsilon/11 \leqq 1/11$, we have $(1+\delta)(1-6\delta)^{-1} \leqq 1+\varepsilon$, so the block basis $((\mathbf{u}_i)_{\varepsilon,\pi})_1^m$ is $(1+\varepsilon)$-symmetric. Thus Theorem 1 follows from the two lemmas.

PROOF OF LEMMA 2. (i) This part is a simple observation. Let $|||\mathbf{a}|||$ stand for $\mathbf{E}_\Omega \|\phi_{\varepsilon,\pi}(\mathbf{a})\|$. Then $|||\cdot|||$ is a 1-symmetric norm on $\mathbf{R}^m$, and by hypothesis $|||\Sigma_1^m \mathbf{e}_i||| \geqq n^{1/p}$. Now let $\mathbf{a} = \Sigma_1^m a_i \mathbf{e}_i$ be a vector in $\mathbf{R}^m$ with $a_i \geqq 0$ for every i. For $1 \leqq k \leqq m$ let $\mathbf{a}^{(k)}$ be the vector $\Sigma_1^m a_i \mathbf{e}_{i+k}$ (where $i+k$ is reduced modulo m). Then by the triangle inequality,

$$|||\mathbf{a}||| \geqq m^{-1}\left|\left|\left|\sum_{k=1}^m \mathbf{a}^{(k)}\right|\right|\right| = m^{-1}\left|\left|\left|\sum_{i=1}^m \left(\sum_{j=1}^m a_j\right)\mathbf{e}_i\right|\right|\right|$$

$$= m^{-1}\|\mathbf{a}\|_1 n^{1/p} = n^{-1/q}\|\mathbf{a}\|_1 h.$$

(ii) The second part of Lemma 2 is certainly the most important step in the proof of Theorem 1.

Let us begin by fixing a vector $\mathbf{a} = \mathbf{a}_i$ for some i. Without loss of generality we may take $\| \mathbf{a} \|_1$ to be 1. We shall also assume that the coordinates of $\mathbf{a}$ satisfy $a_1 \geqq \cdots \geqq a_m \geqq 0$. This is for ease of notation. The proof will clearly be valid in the general case. Let $B_1, \ldots, B_{k+1} \subset [m]$ be defined by

$$B_j = \begin{cases} \{i \in [m] : 2^{-j} < a_i \leqq 2^{-(j-1)}\}, & 1 \leqq j \leqq k, \\ \{i \in [m] : a_i \leqq 2^{-k}\}, & j = k+1, \end{cases}$$

where $k = \log_2(200\, mn^{1/q}/\varepsilon)$.

Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be given by $\mathbf{b}_j = \mathbf{a}\,|_{B_j}$ $(1 \leqq j \leqq k)$. For $(\eta, \sigma) \in \Psi$, we define $\mathbf{b}^j_{\eta,\sigma} = (\mathbf{b}_j)_{\eta,\sigma}$. Clearly $\mathbf{b}^j_{\eta,\sigma} = \mathbf{a}_{\eta,\sigma}\,|_{\sigma(B_j)}$, and the absolute values of the coefficients of $\mathbf{b}^j_{\eta,\sigma}$ lie between 2^{-j} and $2^{-(j-1)}$ when $j \leqq k$, and are at most $\varepsilon/200mn^{1/q}$ when $j = k+1$.

For each $1 \leqq r \leqq k+1$, $(\eta, \sigma) \in \Psi$, define a function $f^r_{\eta,\sigma} : \Omega \to \mathbf{R}$ by

$$f^r_{\eta,\sigma}((\varepsilon, \pi)) = \mathbf{E}[\, \| \phi_{\varepsilon',\pi'}(\mathbf{a}_{\eta,\sigma}) \| \;|\; \phi_{\varepsilon',\pi'}(\mathbf{b}^j_{\eta,\sigma}) = \phi_{\varepsilon,\pi}(\mathbf{b}^j_{\eta,\sigma}), j = 1, \ldots, r].$$

Now, for any fixed (η, σ), the sequence of functions $f^0_{\eta,\sigma}$ $(= \mathbf{E}(\| \phi_{\varphi,\pi}(\mathbf{a}_{\eta,\sigma}) \|))$, $f^1_{\eta,\sigma}, \ldots, f^{k+1}_{\eta,\sigma}$ is a martingale. Note that $f^{k+1}_{\eta,\sigma}(\varepsilon, \pi) = \| \phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma}) \|$, although the expectation is not taken over a singleton subset of Ω. This is because if $\phi_{\varepsilon',\pi'}(\mathbf{b}^j_{\eta,\sigma}) = \phi_{\varepsilon,\pi}(\mathbf{b}^j_{\eta,\sigma})$ for $j = 1, \ldots, k+1$ then $\phi_{\varepsilon',\pi'}(\mathbf{a}_{\eta,\sigma}) = \phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma})$. As it happens the fact that $f^0_{\eta,\sigma}, \ldots, f^{k+1}_{\eta,\sigma}$ is a martingale will not concern us. Instead we are interested in upper bounds for the following two quantities:

(a) the number of *distinct* functions $f^r_{\eta,\sigma}$ for any given $1 \leqq r \leqq k$;

(b) the probability (in Ω) that $f^r_{\eta,\sigma}((\varepsilon, \pi))$ differs substantially from $f^{r-1}_{\eta,\sigma}((\varepsilon, \pi))$ for given r and (η, σ).

The estimate in (a) is simple. If we have (η, σ), (η', σ') such that $\mathbf{b}^j_{\eta,\sigma} = \mathbf{b}^j_{\eta',\sigma'}$ for $j = 1, \ldots, r$, then it is easy to see that $f^r_{\eta,\sigma} \equiv f^r_{\eta',\sigma'}$. But the number of distinct choices of $\mathbf{b}^1_{\eta,\sigma}, \ldots, \mathbf{b}^r_{\eta,\sigma}$ is certainly at most $m(m-1)\cdots(m - \Sigma^r_{j=1} |B_j|) \cdot 2^{\Sigma^r_{j=1}|B_j|}$. So, writing $\beta_j = |B_j|$ $(j = 1, \ldots, k)$ and $\gamma_j = \Sigma^j_{i=1} \beta_i$, we obtain that there are at most $(2m)^{\gamma_r}$ distinct functions $f^r_{\eta,\sigma}$.

We shall use well-known martingale techniques to get an estimate in (b). (See Lemma 4, Corollary 5 and the remarks that follow.) For now we quote a result and show why it is all that is needed to prove the second part of Lemma 2.

The result we quote is that for any $(\eta, \sigma) \in \Psi$, $1 \leqq r \leqq k$ and $\delta_r > 0$,

$$\mathbf{P}_\Omega[f^r_{\eta,\sigma}((\varepsilon, \pi)) - f^{r-1}_{\eta,\sigma}((\varepsilon, \pi)) > \delta_r h n^{-1/q}] < \exp\left(- \frac{\delta_r^2 2^{2(r-1)} h}{8 n^{2/q} \beta_r} \right)$$

and

$$\mathbf{P}_\Omega[f^r_{\eta,\sigma}((\varepsilon,\pi)) - f^{r-1}_{\eta,\sigma}((\varepsilon,\pi)) < -\delta_r hn^{-1/q}] < \exp\left(-\frac{\delta_r^2 2^{2(r-1)}h}{8n^{2/q}\beta_r}\right).$$

Because of the bound given earlier for $\| \mathbf{b}^{k+1}_{\eta,\sigma} \|_\infty$ we also have, for any (ε, π), that

$$f^{k+1}_{\eta,\sigma}((\varepsilon,\pi)) - f^k_{\eta,\sigma}((\varepsilon,\pi)) \leqq (\varepsilon/200)hn^{-1/q} = (\varepsilon/200)hn^{-1/q} \| \mathbf{a} \|_1.$$

Note that the above probabilities are both zero in the case $\beta_r = 0$. As in assertion (iii), take $E(\mathbf{a})$ to be $\mathbf{E}_\Omega(\| \phi_{\varepsilon,\pi}(\mathbf{a}) \|)$. Note that is is the same as $\mathbf{E}_\Omega(\| \phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma}) \|)$ for any $(\eta,\sigma)\in\Psi$, as has already been mentioned.

Now suppose that for some $(\varepsilon,\pi)\in\Omega$ it is true that there exists some (η,σ) for which

$$\| \phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma}) \| - E(\mathbf{a}) > \frac{\varepsilon}{33} \| \mathbf{a} \|_1 hn^{-1/q},$$

i.e., for which

$$f^{k+1}_{\eta,\sigma}((\varepsilon,\pi)) - f^0_{\eta,\sigma}((\varepsilon,\pi)) > \frac{\varepsilon}{33} \| \mathbf{a} \|_1 hn^{-1/q}.$$

Then

$$f^k_{\eta,\sigma}((\varepsilon,\pi)) - f^0_{\eta,\sigma}((\varepsilon,\pi)) > \frac{\varepsilon}{40} \| \mathbf{a} \|_1 hn^{-1/q},$$

so if $\delta_1 + \cdots + \delta_k \leqq \varepsilon/40$, there will be some $1 \leqq r \leqq k$ such that

$$f^r_{\eta,\sigma}((\varepsilon,\pi)) - f^{r-1}_{\eta,\sigma}((\varepsilon,\pi)) > \delta_r \| \mathbf{a} \|_1 hn^{-1/q}.$$

However, by the estimates in (a) and (b) and the normalization $\| \mathbf{a} \|_1 = 1$, the probability of such r and (η,σ) existing is at most

$$\sum_{r=s}^{k} (2m)^{\gamma_r} \exp\left(-\frac{2^{2(r-1)}\delta_r^2 h}{8n^{2/q}\beta_r}\right)$$

where s is the smallest value of r for which $\gamma_r > 0$. This is therefore an upper bound for the probability we wish to estimate in the lemma.

It remains for us to choose appropriate $\delta_1,\ldots,\delta_k$ and to verify that this probability is at most $\frac{1}{2}\cdot m^{-\delta^{-1}\log(3\delta^{-1})}$. Since the other inequality is exactly similar, we will then be done.

Choosing $\delta_r = 2^{-r}\beta_r^{1/2}\gamma_r^{1/2}\cdot\varepsilon/66$ will do.

First,

$$\begin{aligned}\sum_1^k \delta_r &= \frac{\varepsilon}{66}\cdot\sum_1^k 2^{-r}\beta_r^{1/2}\gamma_r^{1/2}\\
&\leqq \frac{\varepsilon}{66}\left(\sum_1^k 2^{-r}\beta_r\right)^{1/2}\left(\sum_1^k 2^{-r}\gamma_r\right)^{1/2} \quad \text{(by the Cauchy–Schwarz inequality)}\\
&\leqq \frac{\varepsilon}{66}\left(\sum_{r=1}^k 2^{-r}\sum_{j=1}^r \beta_j\right)^{1/2} \quad \left(\text{since } \sum_1^k 2^{-r}\beta_r \leqq \sum_1^m a_i = 1 \text{ and } \gamma_r = \sum_{j=1}^r \beta_j\right)\\
&= \frac{\varepsilon}{66}\left(\sum_{j=1}^k \beta_j \sum_{r=j}^k 2^{-r}\right)^{1/2}\\
&< \frac{\varepsilon\sqrt{2}}{66}\left(\sum_{j=1}^k 2^{-j}\beta_j\right)^{1/2} \leqq \frac{\varepsilon}{40}.\end{aligned}$$

Second,

$$\begin{aligned}\sum_{r=s}^k (2m)^{\gamma_r}\exp\left(-\frac{2^{2(r-1)}\delta_r^2 h}{8\beta_r n^{2/q}}\right) &= \sum_{r=s}^k \exp\left(\gamma_r\left(\log(2m) - \frac{\varepsilon^2 h}{4\times 8\times 66^2 n^{2/q}}\right)\right)\\
&\leqq k\exp\left(\log(2m) - \frac{\varepsilon^2 h}{150{,}000 n^{2/q}}\right)\end{aligned}$$

since $\gamma_r > 0$ for every $r \geqq s$.

But since $h \geqq (20/\varepsilon)\cdot(\log(33/\varepsilon))\cdot(150{,}000/\varepsilon^2)\cdot n^{2/q}\log n$, we have

$$\begin{aligned}k\exp\left(\log(2m) - \frac{\varepsilon^2 h}{150{,}000 n^{2/q}}\right) &\leqq k\exp\left(\left(1 - \frac{20}{\varepsilon}\log\left(\frac{33}{\varepsilon}\right)\right)\log n\right)\\
&\leqq \tfrac{1}{2}\exp\left(-\frac{11}{\varepsilon}\log\left(\frac{33}{\varepsilon}\right)\log n\right)\\
&\leqq \tfrac{1}{2}m^{-(11/\varepsilon)\log(33/\varepsilon)}\\
&= \tfrac{1}{2}m^{-\delta\log(3\delta^{-1})}\end{aligned}$$

which is what we needed. This completes the proof of Lemma 2, except for the probabilistic estimate which was quoted, and which will be proved at the end of the section. □

PROOF OF LEMMA 3. We shall prove Lemma 3 by splitting it into a number of further simple lemmas. First let us define two partial orders on $\mathbf{R}^m$. We shall say $\mathbf{a} \leqq_1 \mathbf{b}$ if $\mathbf{a}$ can be written as a convex combination of various $\mathbf{b}_{\eta,\sigma}$, and $\mathbf{a} \leqq_2 \mathbf{b}$ if for all $1 \leqq k \leqq n$, $\Sigma_{i=1}^k a_i^* \leqq \Sigma_{i=1}^k b_i^*$. It is easy to check that these are indeed partial orders. The order $\leqq_1$ can be defined a little more naturally, but with this definition it is immediate that if $\|\cdot\|$ is a 1-symmetric norm on $\mathbf{R}^m$ then $\mathbf{a} \leqq_1 \mathbf{b}$ implies that $\|\mathbf{a}\| \leqq \|\mathbf{b}\|$. This is why we shall use the order. The following easy fact is well known. It was originally noted by Rado [15].

LEMMA 4. *Let $\mathbf{a} = \Sigma_1^m a_i \mathbf{e}_i$ and $\mathbf{b} = \Sigma_1^m b_i \mathbf{e}_i$ be vectors in $\mathbf{R}^m$. Then $\mathbf{a} \leqq_1 \mathbf{b}$ if and only if $\mathbf{a} \leqq_2 \mathbf{b}$.*

For given $\delta > 0$, let us define a subspace $U = U(\delta)$ of $\mathbf{R}^m$, as follows. Let $r = \lfloor \log_{1+\delta} m \rfloor$, and for $1 \leqq i \leqq r$ set $k_i = \lfloor (1+\delta)^i \rfloor$ and $\mathbf{u}_i = \Sigma_{j=1}^{k_i} \mathbf{e}_j$. Let U be the subspace generated by $\mathbf{u}_1, \ldots, \mathbf{u}_r$. For technical reasons, let k_{r+1} be defined to be $n+1$. Note that for $1 \leqq i \leqq r$, $k_{i+1} - 1 \leqq (1+\delta)k_i$. In the next lemma we show that any 1-symmetric norm is determined to within a factor $1+\delta$ by its restriction to U. We shall eventually need a perturbation of this result.

For the next three lemmas, let $\delta > 0$ and let $U = U(\delta)$ be as defined above.

LEMMA 5. *Let $|||\cdot|||$ be a 1-symmetric norm which is fixed on U. Then it is determined to within a factor of $(1+\delta)$ on $\mathbf{R}^m$.*

PROOF. Let $\mathbf{a} = \Sigma_1^m a_i \mathbf{e}_i$ and let $a_1 \geqq \cdots \geqq a_m \geqq 0$. Let us define two vectors $\mathbf{a}'$ and $\mathbf{a}''$ in U by

$$\mathbf{a}' = \sum_{i=1}^{r} (a_{k_i} - a_{k_{i+1}}) \sum_{j=1}^{k_{i+1}-1} \mathbf{e}_j = \sum_{i=1}^{r} a_{k_i} \sum_{j=k_i}^{k_{i+1}-1} \mathbf{e}_j$$

and

$$\mathbf{a}'' = \sum_{i=1}^{r} (a_{k_i} - a_{k_{i+1}}) \sum_{j=1}^{k_i} \mathbf{e}_j = \sum_{i=1}^{r} a_{k_i} \sum_{j=k_{i-1}+1}^{k_i} \mathbf{e}_j.$$

Then $\mathbf{a}''$ is dominated pointwise by $\mathbf{a}$, which is dominated pointwise by $\mathbf{a}'$. Thus $|||\mathbf{a}''||| \leqq |||\mathbf{a}||| \leqq |||\mathbf{a}'|||$. We shall show that $(1+\delta)\mathbf{a}'' \geqq_2 \mathbf{a}'$ and hence (by Lemma 4) that $(1+\delta)|||\mathbf{a}''||| \geqq |||\mathbf{a}'|||$, which will complete the proof. Let us write a_j' and a_j'' for the coordinates of $\mathbf{a}'$ and $\mathbf{a}''$ respectively. Then for any $1 \leqq s \leqq m$

$$\sum_{j=1}^{s} a_j' = \sum_{i=1}^{r} (a_{k_i} - a_{k_{i+1}}) \min\{s, k_{i+1} - 1\}$$

and

$$\sum_{j=1}^{s} a_j'' = \sum_{i=1}^{r} (a_{k_i} - a_{k_{i+1}}) \min\{s, k_i\}.$$

But for each i, as remarked above, $k_{i+1} - 1 \leqq (1+\delta)k_i$, so clearly $\min\{s, k_{i+1} - 1\} \leqq (1+\delta)\min\{s, k_i\}$. Hence $\mathbf{a}' \leqq_2 (1+\delta)\mathbf{a}''$ as stated. □

LEMMA 6. *Let $\delta' > 0$ and let $\|\cdot\|$ be any norm on $\mathbf{R}^m$ which is $(1-\delta')^{-1}$-symmetric at any $\mathbf{a} \in U(\delta)$. Then $\|\cdot\|$ is $(1+\delta)(1-2\delta')^{-1}$-symmetric on $\mathbf{R}^m$.*

PROOF. Define a norm $|||\cdot|||$ on $\mathbf{R}^m$ by $|||\mathbf{a}||| = \max\{\|\mathbf{a}_{\eta,\sigma}\| : (\eta, \sigma) \in \Psi\}$. Then if $\mathbf{a} \in U$, we have $\|\mathbf{a}\| \leqq |||\mathbf{a}||| \leqq (1+\delta)\|\mathbf{a}\|$ by assumption. Let $\mathbf{a} \in \mathbf{R}^m$, assume, for the sake of simplicity of the exposition, that $a_1 \geqq \cdots \geqq a_m \geqq 0$ and define $\mathbf{a}''$ as in the proof of Lemma 5. We know that $\|\mathbf{a}\| \leqq |||\mathbf{a}|||$. In the other direction, since $\mathbf{a}''$ is dominated pointwise by $\mathbf{a}$, $2\mathbf{a}'' - \mathbf{a}$ is dominated pointwise by $\mathbf{a}''$, so $|||2\mathbf{a}'' - \mathbf{a}||| \leqq |||\mathbf{a}''|||$. But since $\mathbf{a}'' \in U$, we have

$$\begin{aligned} \|\mathbf{a}\| &\geqq 2\|\mathbf{a}''\| - \|2\mathbf{a}'' - \mathbf{a}\| \\ &\geqq 2(1-\delta')|||\mathbf{a}''||| - |||\mathbf{a}''||| \\ &= (1-2\delta')|||\mathbf{a}''||| \geqq (1-2\delta')(1+\delta)^{-1}|||\mathbf{a}||| \end{aligned}$$

where the last inequality follows from the proof of Lemma 5.

Hence $\|\cdot\|$ is $(1+\delta)(1-2\delta')^{-1}$-equivalent to the 1-symmetric norm $|||\cdot|||$, which proves the lemma. □

LEMMA 7. *Let $\|\cdot\|$ be a norm on $\mathbf{R}^m$, and define $|||\cdot|||$ as in the proof of Lemma 6. Suppose that the set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ forms a δ-net in $|||\cdot|||$ of the $|||\cdot|||$-unit ball of U and that $\|\cdot\|$ is $(1+\delta)$-symmetric at every $\mathbf{a}_i$. Then $\|\cdot\|$ is $(1+\delta)(1-6\delta)^{-1}$-symmetric on $\mathbf{R}^m$.*

PROOF. By hypothesis, given any $1 \leqq i \leqq N$ and any $(\eta, \sigma) \in \Psi$, we have $\|(\mathbf{a}_i)_{\eta,\sigma}\| \geqq (1+\delta)^{-1}|||\mathbf{a}_i|||$. Let us pick $\mathbf{a} \in U$ with $|||\mathbf{a}||| = 1$. We shall show that $\|\cdot\|$ is $(1-3\delta)^{-1}$-symmetric at $\mathbf{a}$. So pick any $(\eta, \sigma) \in \Psi$ and pick i such that $|||\mathbf{a} - \mathbf{a}_i||| \leqq \delta$. Write $\mathbf{b} = \mathbf{a}_{\eta,\sigma}$ and $\mathbf{b}' = (\mathbf{a}_i)_{\eta,\sigma}$. Then we have

$$\begin{aligned} |||\mathbf{b}||| - \|\mathbf{b}\| &\leqq |||\mathbf{b} - \mathbf{b}'||| + \big|\, |||\mathbf{b}'||| - \|\mathbf{b}'\| \,\big| + \|\mathbf{b}' - \mathbf{b}\| \\ &\leqq \delta + 1 - (1+\delta)^{-1} + \delta < 3\delta. \end{aligned}$$

But $|||\mathbf{b}||| = |||\mathbf{a}||| = 1$, so the norm $\|\cdot\|$ is indeed $(1-3\delta)^{-1}$-symmetric at every vector in U. But then, by Lemma 6, it is $(1+\delta)(1-6\delta)^{-1}$-symmetric on $\mathbf{R}^m$, as stated. □

We are now in a position to prove Lemma 3. The dimension of $U(\delta)$ is most $\log_{1+\delta} m$, so a standard estimate (see e.g. [13]) tells us that there is a δ-net of the $|||\cdot|||$-unit ball of U of cardinality at most $(1 + 2/\delta)^{\log_{1+\delta} m}$. When $\delta \leqq 1/11$ one can easily check that this is at most $m^{\delta^{-1}\log(3\delta^{-1})}$. Lemma 3 now follows immediately from Lemma 7. □

We have now proved Theorem 1, with the exception of the probabilistic estimate which we quoted earlier for the estimate (b) in the proof of the second part of Lemma 2. This estimate is based on the following standard martingale inequality which can be found in many places in the literature, for example in [13]. This inequality was first used in the local theory of Banach spaces by Maurey [12], and the method was then developed by Schechtman [16].

LEMMA 8. *Let $f_0 = \mathbf{E}f, f_1, \ldots, f_n = f$ be a martingale with a difference sequence $d_i = f_i - f_{i-1}$ which satisfies $\| d_i \|_\infty \leqq c_i$ for $1 \leqq i \leqq n$. Then, for any $a > 0$,*

$$\mathbf{P}[f - \mathbf{E}f \geqq a] \leqq \exp\left(-\frac{a^2}{2 \sum_{i=1}^n c_i^2} \right). \qquad \square$$

Let $(\Phi, d, \mathbf{P})$ be the metric probability space $\{-1, 1\}^n \times S_n$, where

$$d(\varepsilon, \pi), (\varepsilon', \pi')] = \sum_{i=1}^n \{b_i : \varepsilon_i \neq \varepsilon_i' \text{ or } \pi(i) \neq \pi'(i)\}$$

for a sequence $b_1 \geqq \cdots \geqq b_n \geqq 0$, and the measure $\mathbf{P}$ is the normalized counting measure on Φ. Define equivalence relations $\sim_0, \ldots, \sim_n$ on Φ by $(\varepsilon, \pi) \sim_i (\varepsilon', \pi')$ iff $\varepsilon_j = \varepsilon_j'$ and $\pi(j) = \pi'(j)$ for $1 \leqq j \leqq i$. For $1 \leqq i \leqq n$ let $\mathscr{F}_i$ be the sigma-field whose atoms are the equivalence classes of $\sim_i$. Finally, let f be a 1-Lipschitz function on Φ, and set $f_i = \mathbf{E}(f \mid \mathscr{F}_i)$ $(1 \leqq i \leqq n)$.

The next corollary is a simple deduction from Lemma 8. It is proved in detail in [7].

COROLLARY 9. *Let $(\Phi, d, \mathbf{P})$ and $f_0, \ldots, f_n$ be defined as above. Then for all $s > t$ and $\delta > 0$,*

$$\mathbf{P}[f_s - f_t \geqq \delta] \geqq \exp\left(-\frac{\delta^2}{8 \sum_{t+1}^s b_i^2} \right)$$

and

$$\mathbf{P}[f_s - f_t \leqq -\delta] \leqq \exp\left(-\frac{\delta^2}{8 \sum_{t+1}^s b_i^2} \right). \qquad \square$$

Setting $\Phi = \Omega$, $b_i = a_{\lceil (i/h) \rceil}$ $(1 \leqq i \leqq n)$ and $s = \gamma_r h$, $t = \gamma_{r-1} h$, we have $2^{-r} \leqq b_i \leqq 2^{-(r-1)}$ for $\gamma_{r-1} h \leqq i \leqq \gamma_r h$. Set $f((\varepsilon, \pi)) = \| \phi_{\varepsilon,\pi}(\mathbf{a}_{\eta,\sigma}) \|$.

Since

$$(\varepsilon, \pi) \sim_s (\varepsilon', \pi') \Rightarrow \phi_{\varepsilon,\pi}(\mathbf{b}_{\eta,\sigma}^j) = \phi_{\varepsilon',\pi'}(\mathbf{b}_{\eta,\sigma}^j) \qquad (1 \leqq j \leqq r)$$

and

$$(\varepsilon,\pi)\sim_t(\varepsilon',\pi')\Rightarrow\phi_{\varepsilon,\pi}(\mathbf{b}^j_{\eta,\sigma})=\phi_{\varepsilon',\pi'}(\mathbf{b}^j_{\eta,\sigma})\qquad(1\leqq j\leqq r-1)$$

and f is 1-Lipschitz, we obtain from Corollary 9 that

$$\mathbf{P}[f^r_{\eta,\sigma}((\varepsilon,\pi))-f^{r-1}_{\eta,\sigma}((\varepsilon,\pi))>\delta_r hn^{-1/q}]<\exp\left(-\frac{\delta_r^2h^2n^{-2/q}}{8(\gamma_r-\gamma_{r-1})2^{-2(r-1)}h}\right)$$

$$=\exp\left(-\frac{2^{2(r-1)}\delta_r^2h}{8n^{2/q}\beta_r}\right).$$

Similarly,

$$\mathbf{P}[f^r_{\eta,\sigma}((\varepsilon,\pi))-f^{r-1}_{\eta,\sigma}((\varepsilon,\pi))<-\delta_r hn^{-1/q}]\leqq\exp\left(-\frac{2^{2(r-1)}\delta_r^2h}{8n^{2/q}\beta_r}\right).$$

This is the estimate that was assumed earlier. The proof of Theorem 1 is now complete.

§3. The upper bound

Discussion of the upper bound is somewhat complicated by the fact that there can often be block bases which are almost symmetric but which do not satisfy certain other very natural conditions. The following example shows that in one sense the result of the last section is best possible.

Let $1\leqq p<2$, $1/q+1/p=1$, $m=2n^{2/p-1}$ and $h=n^{2/q}/2$. Then let $\mathbf{e}_1,\dots,\mathbf{e}_m$ be the standard basis of l_1^m. For $1\leqq i\leqq n$ we set $x_i=\mathbf{e}_{k(i)}$, where $k(i)=\lfloor i/H\rfloor$. Then

$$\mathbf{E}\left\|\sum_1^n\varepsilon_ix_i\right\|=\mathbf{E}\left(\sum_{i=1}^m\left|\sum_{j=1}^h\varepsilon_{(h-1)i+j}\right|\right)$$

$$=m\mathbf{E}\left|\sum_{j=1}^h\varepsilon_j\right|$$

$$\geqq 2^{-1/2}mh^{1/2}=n^{1/p}.$$

It is immediate by linear algebra that the sequence $(x_i)_1^n$ has no almost symmetric block basis of cardinality greater than m. Unfortunately, however, this sequence is not linearly independent. The natural way of dealing with this ought to be simply to embed l_1^m into any Banach space of dimension n and perturb the sequence very slightly, but this does not quite work. Suppose, for example, that we were to embed l_1^m into l_1^n in the natural way and then perturb the unit vector basis of l_1^m by adding multiples of vectors from the unit vector basis of l_1^n. We would then be able to construct a proportional-sized block basis that was 1-symmetric by taking blocks of the form $x_i - x_{i+1}$ (where i should not be a multiple of h) and normalizing. Although this is a contrived example, it illustrates the point that to find a linearly independent sequence with average growth as given in the main theorem, but with no almost symmetric block sequence, either one must use a completely different example, or one must solve the rather different problem of constructing some independent sequence with more or less no almost symmetric block sequence of any size. As p approaches 2, these become the same problem.

The 1-symmetric block basis given in the example above is, however, an uninteresting one. The reason is that the normalization can involve multiplying by an arbitrarily large number, so that the block basis bears very little relation to the original basis. If we pick any real number M and stipulate first that the coefficients appearing in the vectors in the block basis should be bounded above by M, and second that the block basis should be normalized, then any sufficiently small perturbation of the linearly dependent sequence given above will clearly do. This is obviously a natural restraint to impose on the block basis we find.

We shall present a randomly constructed space which gives a bound close to that achieved in the other direction in the last section without this extra condition when $1 \leqq p \leqq 3/2$ and which also satisfies a lower p-estimate. When $p > 3/2$ this construction gives a basis with a large block basis that is close to the unit vector basis of an appropriate l_∞^k. Since a detailed proof that the space does indeed give such a bound is rather long, we shall describe the construction and only give a sketch of the proof.

THEOREM 10. *There exists an absolute constant C such that for any $1 \leqq p \leqq 3/2$ and any $n \in \mathbf{N}$, there is a norm $\|\cdot\|$ on $\mathbf{R}^n$ satisfying the following three conditions*:

(i) *the standard basis is normalized*;

(ii) *for any $\mathbf{a} \in \mathbf{R}^n$, $\|\mathbf{a}\| \geqq \|\mathbf{a}\|_p$*;

(iii) *if $k \geqq Cn^{2/p-1}(\log n)^{4/3}$ then no block basis $\mathbf{u}_1, \ldots, \mathbf{u}_k$ of the standard basis is 2-symmetric.*

PROOF (sketch). Set $m = (C/2)n^{2/p-1}(\log n)^{1/3} = k/2 \log n$, $h = (2/C)n^{2/q}(\log n)^{-4/3} = 2n/k$, $N = 600h^{2/p} \log n$ and $M = (5n^2)^{mh}$. Pick N functionals $f_1, \ldots, f_N$ independently at random from the set of all functionals on $\mathbf{R}^n$ with $\pm$ 1-coordinates, where this set is endowed with the uniform probability measure. Given a vector $\mathbf{a} \in \mathbf{R}^n$, set

$$\|\mathbf{a}\| = \max\{|f_1(\mathbf{a})|, \ldots, |f_N(\mathbf{a})|, \|\mathbf{a}\|_p\}.$$

Given any block basis $\mathbf{u}_1, \ldots, \mathbf{u}_m$, the probability that we can find some $1 \leqq j \leqq m$ and some sequence of signs $\eta_1, \ldots, \eta_m$ such that

$$(*) \qquad \left\|\sum_1^m \eta_i \mathbf{u}_i\right\| \geqq \max\left\{80 \log n \,\|\mathbf{u}_j\|_1, 6 \left\|\sum_1^m \eta_i \mathbf{u}_i\right\|_p\right\}$$

is greater than $1 - M^{-1}$. We shall indicate how this is proved. Let us say in this case that the block basis $(\mathbf{u}_i)_1^m$ satisfies condition (*). We shall say that it satisfies condition (**) if there are $1 \leqq j \leqq m$ and a sequence of signs $\eta_1, \ldots, \eta_m$ such that

$$(**) \qquad \left\|\sum_1^m \eta_i \mathbf{u}_i\right\| \geqq \max\left\{64 \log n \,\|\mathbf{u}_j\|, 4 \left\|\sum_1^m \eta_i \mathbf{u}_i\right\|_p\right\}.$$

Let us now fix a block basis $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Since the supports of the $\mathbf{u}_i$ are disjoint, we can find $k/2$ of them such that each one is supported on at most h coordinates. Passing to a subsequence again, and relabelling, we can find a subbasis $\mathbf{u}_1, \ldots, \mathbf{u}_m$ and $\lambda > 0$ such that for each $1 \leqq i \leqq m$, $\lambda \leqq \|\mathbf{u}_i\|_p \leqq 2\lambda$. By multiplying each element of the basis by λ^{-1} we may assume that $\lambda = 1$. Let us call such a block basis *proper*. Note that if this proper block basis fails to be 2-symmetric, then the block basis $(\mathbf{u}_i)_1^k$ also fails.

We now fix a particular $1 \leqq j \leqq N$ and examine the effect of the functional f on the space generated by $\mathbf{u}_1, \ldots, \mathbf{u}_m$.

There certainly exist signs $\eta_1, \ldots, \eta_m$ such that $f(\Sigma_1^m \eta_i \mathbf{u}_i) = \Sigma_1^m |f(\mathbf{u}_i)|$. Further, it is a fairly straightforward application of Lemma 8 that there exists an absolute constant c such that for each $\mathbf{u}_i$, we have

$$\mathbf{P}[|f(\mathbf{u}_i)| \geqq c \,\|\mathbf{u}_i\|_2] \geqq 1/2.$$

We can therefore define some independent random variables $\gamma_1, \ldots, \gamma_m$ in such a way that for each i, γ_i is dominated by $|f(\mathbf{u}_i)|$, and takes the values $c \,\|\mathbf{u}_i\|_2$ or 0, each with probability 1/2. Then $\mathbf{E}(\Sigma_1^m \gamma_i) = \frac{1}{2} c \,\Sigma_1^m \|\mathbf{u}_i\|_2$. By an easy application of Lemma 8 and the fact that each $\mathbf{u}_i$ is supported on at most h coordinates, one can show that

$$\mathbf{P}\left[\sum_1^m \gamma_i \leqq \frac{c}{4} \sum_1^m \|\mathbf{u}_i\|_2\right] \leqq \exp\left(-\frac{c}{64} m h^{1-2/p}\right).$$

Let us set $\theta = \exp(-\frac{1}{64} c m h^{1-2/p})$.

Using the fact that the support of each $\mathbf{u}_i$ is at most h, that the $\| \mathbf{u}_i \|_p$ differ by a factor of at most 2 and that $p \leqq 3/2$, one can also show that

$$\sum_1^m \| \mathbf{u}_i \|_2 \geqq \max \left\{ \frac{C^{1/2}}{2^{3/2}} (\log n)^{1/p + 1/3} \left\| \sum_1^m \mathbf{u}_i \right\|_p, (C/2)^{3/2} \log n \, \| \mathbf{u}_j \|_1 \right\}.$$

It follows that if $(c/4)\cdot(C/2)^{3/2} > 80$, i.e., if $C > 2\cdot(320/c)^{2/3}$, then

$$\mathbf{P}\left[\sum_1^m |f(\mathbf{u}_i)| < \max \left\{ 80 \log n \, \| \mathbf{u}_j \|_1, 6 \left\| \sum_1^m \mathbf{u}_i \right\|_p \right\} \right] \leqq \theta.$$

Moreover, since the functionals $f_1, \ldots, f_N$ were chosen independently, the probability that for every j and every choice of signs $\eta_1, \ldots, \eta_m$,

$$\left\| \sum_1^m \eta_i \mathbf{u}_i \right\| < \max \left\{ 80 \log n \, \| \mathbf{u}_j \|_1, 6 \left\| \sum_1^m \mathbf{u}_i \right\|_p \right\},$$

is at most $\theta^N = \exp(-\frac{1}{64} cmh^{1-2/p} N)$. In other words, any given proper block basis satisfies condition (*) with probability at least $1 - \theta^N$. One can check that this is at least $1 - M^{-1}$.

Next, given $\delta > 0$, let us define two block bases $(\mathbf{u}_i)_1^m$ and $(\mathbf{u}_i')_1^m$ to be δ-close if they are related in the following way. First, for any $i \neq j$, $\mathrm{supp}(\mathbf{u}_i) \cap \mathrm{supp}(\mathbf{u}_j') = \emptyset$, and second, for each i, $\| \mathbf{u}_i - \mathbf{u}_i' \|_p \leqq \delta$. We must estimate, for given δ, the size required of a set of block bases for every proper block basis of cardinality k to be δ-close to at least one block basis in the set. Since the number of ways of choosing m disjoint sets of size h is at most n^{mh}, and the number of vectors in a δ-net of the 2-ball of l_p^h is at most $(1 + 4/\delta)^h$ [13], we find that the size needed is certainly no greater than $(5n/\delta)^{mh} = M$.

Now fix δ to be n^{-1}, let $(\mathbf{u}_i)_1^m$ be a proper block basis, let $(\mathbf{v}_i)_1^m$ be δ-close to $(\mathbf{u}_i)_1^m$ and suppose that $(\mathbf{v}_i)_1^m$ satisfies condition (*). Since $(\mathbf{u}_i)_1^m$ is a proper block basis, $\| \Sigma_1^m \eta_i \mathbf{u}_i \| \geqq m^{1/p}$ for every choice of signs $(\eta_i)_1^m$. It is straightforward to show that in this case $(\mathbf{u}_i)_1^m$ satisfies condition (**). We have therefore shown that there exists a choice of functionals $f_1, \ldots, f_N$ such that every proper block basis satisfies condition (**). Since every block basis of cardinality k has a subbasis, a multiple of which is proper, it remains only to show that no block basis satisfying condition (**) is 2-symmetric.

Suppose then that a block basis $(\mathbf{u}_i)_1^m$ satisfies condition (**) and is 2-symmetric. Then for *all* choices of sign $(\eta_i)_1^m$ and any j, we have

$$\left\| \sum_1^m \eta_i \mathbf{u}_i \right\| \geqq \max \left\{ 16 \log n \, \| \mathbf{u}_j \|_1, 2 \left\| \sum_1^m \eta_i \mathbf{u}_i \right\|_p \right\}.$$

It follows that each vector of the form $\Sigma_1^m \eta_i \mathbf{u}_i$ is supported by one of $f_1, \ldots, f_N$. One can show that, on the contrary, no given f_j can support as many as $N^{-1}2^m$ of these vectors. Let us fix $1 \leqq j \leqq N$ and set $f = f_j$. Since the block basis $(\mathbf{u}_i)_1^m$ is 2-symmetric, there exists some $\mu > 0$ such that for every choice of signs $(\eta_i)_1^m$,

$$\mu \| \mathbf{u}_1 \| \leqq \left\| \sum_1^m \eta_i \mathbf{u}_i \right\| \leqq 2\mu \| \mathbf{u}_1 \| ,$$

from which it follows that $\Sigma_1^m | f(\mathbf{u}_i) | \leqq 2\mu \| \mathbf{u}_1 \|$. We know also that $\mu \geqq 16 \log n$ and that, for each i, $| f(\mathbf{u}_i) | \leqq 2 \| \mathbf{u}_1 \|$. Let Ω be the probability space of all sequences of signs $(\eta_i)_1^m$ uniformly distributed. By using these facts and Lemma 8 again, one can show that

$$\mathbf{P}\left[\left| f\left(\sum_1^m \eta_i \mathbf{u}_i \right) \right| \geqq \mu \| \mathbf{u}_1 \| \right] \leqq 2 \exp\left(-\frac{1}{4} \left(\sum_1^m | f(\mathbf{u}_i) | \right)^2 \Big/ 2 \sum_1^m | f(\mathbf{u}_i) |^2 \right)$$

$$\leqq 2 \exp(-\log n) < N^{-1}.$$

This contradiction completes the proof of Theorem 10. □

Although this theorem applies only when $p \leqq 3/2$, the same construction and proof show that in the other cases we can ensure that any symmetric block basis has only very limited growth.

In the light of Theorem 10 it is natural to ask the following question. Suppose we have a normalized sequence $x_1, \ldots, x_n$ which satisfies a lower 2-estimate. That is, for any sequence of scalars $(a_i)_1^n$, suppose that

$$\left\| \sum_1^n a_i x_i \right\| \geqq \left(\sum_1^n a_i^2 \right)^{1/2} .$$

Then how large a $(1 + \varepsilon)$-symmetric block sequence can be guaranteed to exist? By considering the case $p = 3/2$ in Theorem 10, we find that we cannot expect to find a block sequence of size greater than $n^{1/3}$. However, it is almost certainly the case that for any $\alpha > 0$ and any n_0, there is some $n \geqq n_0$ and a basis $x_1, \ldots, x_n$ which satisfies the above conditions but has no $(1 + \varepsilon)$-symmetric block basis of cardinality n^α.

§4. Type conditions

As was mentioned in the introduction, Amir and Milman [1] have discussed the problem of finding an almost symmetric block basis when the original basis is extremal in a space with some type constant. More specifically, suppose the sequence in Theorem 1 is in a Banach space X with p-type constant C. Does this enable us to find an almost symmetric block sequence of cardinality substantially greater than $n^{2/p-1}$? In this section we show that if p is close to or equal to 2, and the type constant C of X is fairly close to 1, then there is a $(1+\varepsilon)$-symmetric block basis of cardinality at least $\alpha(\varepsilon)n^{\gamma}$, for an exponent γ that exceeds $2/p-1$. Our method involves obtaining estimates of cotype constants in terms of type constants.

First, let us consider the case $p=2$. Here, any result at all that gives a power of n is interesting, since no such bound follows from the result of Section 1 or from results of Amir and Milman. We shall obtain a block sequence that is $(1+\varepsilon)$-symmetric and whose cardinality is a power of n that depends on the precise type constant, tending to 1/2 as the constant tends to 1. We shall of course insist that the original basis is independent, since the 1-dimensional Banach space has 2-type constant 1. This in fact shows that a purely probabilistic method cannot hope to work. In the next simple lemma, we show how to replace the original basis by a block basis that looks much nicer. Lemma 12 shows that a space with a good type-2 constant cannot have too large a 2-cotype constant.

LEMMA 11. *Let $x_1,\dots,x_n$ be a linearly independent sequence of vectors in l_2. Then there is an orthonormal block sequence $y_1,\dots,y_m$ of $x_1,\dots,x_n$ of cardinality $m=\sqrt{2n}$.*

PROOF. Set $y_1=x_1/\|x_1\|$. Now there is at least one unit vector in the space spanned by x_2 and x_3 which is orthogonal to y_1. Let y_2 be such a vector. There is at least one unit vector in the space spanned by x_4, x_5 and x_6 which is orthogonal to both y_1 and y_2. Let y_3 be such a vector. Continuing this process gives an orthonormal block basis of cardinality m, where m is the greatest integer such that $\frac{1}{2}m(m-1)\leqq n$. In particular, $m\geqq\sqrt{2n}$. □

LEMMA 12. *There exists an absolute constant γ with the following property. Let $0\leqq\varepsilon\leqq 1$ and let X be any n-dimensional Banach space satisfying $T_2(X)\leqq 1+\varepsilon$. Then*

$$C_2(X)\leqq\gamma n^{\log(1+\varepsilon)/\log 2}\log n\leqq\gamma n^{\varepsilon/\log 2}\log n.$$

PROOF. In fact, we use a rather weaker hypothesis. For any $x_1,x_2\in X$ we have

$$\tfrac{1}{2}(\|x_1+x_2\|^2+\|x_1-x_2\|^2)\leqq(1+\varepsilon)^2(\|x_1\|^2+\|x_2\|^2).$$

By a simple substitution this implies that for any $y_1, y_2 \in X$

$$(*) \qquad \tfrac{1}{2}(\|y_1+y_2\|^2+\|y_1-y_2\|^2) \geqq (1+\varepsilon)^{-2}(\|y_1\|^2+\|y_2\|^2).$$

König and Tzafriri [9] have shown that there is an absolute constant γ_0 such that for any n-dimensional space Y,

$$C_2(Y) \leqq \gamma_0 C_2(Y, n)(\log n)^{1/2}.$$

Let $N = 2^k$ be the smallest power of 2 that is greater than n, so $k \leqq 1 + \log n/\log 2$, and suppose we have a sequence of vectors $x_1, \dots, x_N \in X$. By repeated applications of $(*)$ we find that

$$\begin{aligned} \mathbf{E}\left\|\sum_1^N \varepsilon_i x_i\right\|^2 &\geqq (1+\varepsilon)^{-2}\left(\mathbf{E}\left\|\sum_1^{N/2} \varepsilon_i x_i\right\|^2 + \mathbf{E}\left\|\sum_{N/2+1}^{N} \varepsilon_i x_i\right\|^2\right) \\ &\geqq \cdots \\ &\geqq (1+\varepsilon)^{-2k}\sum_1^N \|x_i\|^2. \end{aligned}$$

We therefore have

$$\begin{aligned} C_2(X, n) \leqq C_2(X, N) \leqq (1+\varepsilon)^k &\leqq (1+\varepsilon)\cdot(1+\varepsilon)^{\log_e n/\log_e 2} \\ &= (1+\varepsilon)n^{\log(1+\varepsilon)/\log 2} \leqq 2n^{\varepsilon/\log 2}. \end{aligned}$$

Thus, setting $\gamma = 2\gamma_0$, we have $C_2(X) \leqq \gamma n^{\varepsilon/\log 2}(\log n)^{1/2}$. □

Theorem 13. *Suppose $x_1, \dots, x_n$ is a linearly independent sequence of vectors spanning a Banach space X with 2-type constant $1+\varepsilon$. Then there is a $(1+\varepsilon)$-symmetric block sequence $y_1, \dots, y_m$ of $x_1, \dots, x_n$ of cardinality at least $m = \alpha n^{f(\varepsilon)}/(\log n)^6$, where $f(\varepsilon) = \frac{1}{2} - 5\varepsilon/\log 2$ and $\alpha = \alpha(\varepsilon)$.*

Proof. By Lemma 12, X has 2-cotype constant at most $C' = \gamma n^{\varepsilon/\log 2}\log n$. A well known result of Kwapien [10] states that for any k-dimensional Banach space Y,

$$d(Y, l_2^k) \leqq T_2(Y)C_2(Y).$$

Hence, X is actually $C = (1+\varepsilon)C'$-equivalent to l_2^n. Let $T: X \to l_2^n$ be a map such that $\|T\| = 1$ and $\|T^{-1}\| \leqq C$. By Lemma 11 we can pick a block sequence $z_1, \dots, z_m$ with $m = \sqrt{2n}$ such that $Tz_1, \dots, Tz_m$ is an orthonormal sequence. This implies that $z_1, \dots, z_m$ is C-equivalent to the unit vector basis of l_2^m. But Theorem 1 of [7] states that if $1 \leqq p < \infty$, $C > 1$, $0 < \varepsilon < 1/2$ and

$y_1, \dots, y_n$ is a basis which is C-equivalent to the unit vector basis of l_p^n, then $(y_i)_1^n$ has a $(1+\varepsilon)$-symmetric block basis of cardinality at least $\beta(\varepsilon, p, C)n/\log n$, where $\beta = C^{-2p+1}\log C\beta'(e, p)$. By using Lemma 3 of this paper in the place of Lemmas 2 and 3 of [7], this may easily be improved to $\beta = C^{-2p+1}\beta'(\varepsilon, p)$. It follows here that there is a $(1+\varepsilon)$-symmetric block basis of $z_1, \dots, z_m$ of cardinality at least $\alpha(\varepsilon)C^{-5}m/\log m$, that is, at least $\alpha(\varepsilon)n^{f(\varepsilon)}/(\log n)^6$. □

We shall now look at what happens when $p < 2$. It turns out that the proof of Theorem 13 can be adapted easily to give a result here, provided p is sufficiently close to 2 and C is sufficiently close to 1. Let X be an n-dimensional space with p-type constant $C = T_p(X)$. Then we have

$$\tfrac{1}{2}(\|x_1+x_2\|^2 + \|x_1-x_2\|^2) \leqq C^2(\|x_1\|^p + \|x_2\|^p)^{2/p}$$
$$\leqq 2^{2/p-1}C^2(\|x_1\|^2 + \|x_2\|^2).$$

Hence, for any $y_1, y_2 \in X$, we have

$$(**) \qquad \tfrac{1}{2}(\|y_1+y_2\|^2 + \|y_1-y_2\|^2) \geqq 2^{2-2/p}C^{-2}(\|y_1\|^2 + \|y_2\|^2).$$

THEOREM 14. *Let $0<\varepsilon\leqq 1/2$, $1\leqq p<2$, $C\geqq 1$ and let $x_1, \dots, x_n$ be any sequence of vectors spanning an n-dimensional Banach space X which satisfies $T_p(X)\leqq C$. Then $(x_i)_1^n$ has a $(1+\varepsilon)$-symmetric block basis of cardinality at least $\alpha(\varepsilon)n^{f(C,p)}/(\log n)^6$ where $f(C, p) = (1/2) - 5((\log C/\log 2) + (2/p - 1))$.*

PROOF. By using the inequality (**) in place of (*) in the proof of Lemma 12 we obtain that $C_2(X) \leqq \gamma n^{\log(2^{1/p-1/2}C)/\log 2}\log n$. Now a result of Tomczak-Jaegermann [17] states that for any n-dimensional space X, and any $p \leqq 2$ and $q \geqq 2$,

$$d(X, l_2^n) \leqq 4\alpha_p(X)\beta_q(X).$$

Moreover, it is well known that $\alpha_p(X)$ and $T_p(X)$ are the same, up to a constant, and also that $\beta_p(X) \leqq C_p(X)$. So in this case we have

$$d(X, l_2^n) \leqq C\gamma n^{\log(2^{1/p-1/2}C)/\log 2 + 1/p - 1/2}\log n = C\gamma n^{(\log C/\log 2)+(2/p-1)}\log n.$$

Just as in the proof of Theorem 13, we may now apply Lemma 11 and Theorem 1 of [7] to obtain that if $x_1, \dots, x_n$ is any linearly independent sequence of vectors in a space X with $T_p(X) = C$, then it has a $(1+\varepsilon)$-symmetric block basis of cardinality at least $\alpha(\varepsilon)n^{f(C,p)}/(\log n)^6$, as stated. This improves on the bound of $n^{2/p-1}$ obtained in Section 2 if $6(2/p-1) + 5\log C/\log 2 < 1/2$. □

REMARKS. It is possible to demonstrate that if X is an n-dimensional space, then $C_2(X) \leqq \gamma C_2(X, n)\log n$ using well-known results about the duality between type and cotype and a result of Tomczak-Jaegermann about the relationship between $\alpha_p(X)$ and $\alpha_p(X, n)$. It is known (cf. [13] Chapters 9 and 14) that for any n-dimensional normed space X, $C_2(X, n) \geqq (\log n)^{-1}T_2(X^*, n)$, and Tomczak-Jaegermann's result [17] states that $\alpha_p(X, n)$ and $\alpha_p(X)$ are the same, to within an absolute constant. We obtain

$$\begin{aligned} C_2(X, n) &\geqq (\log n)^{-1}T_2(X^*, n) \geqq (\log n)^{-1}\alpha_2(X^*, n) \\ &\geqq \gamma_1(\log n)^{-1}\alpha_2(X^*) \geqq \gamma_3(\log n)^{-1}T_2(X^*) \\ &\geqq \gamma_4(\log n)^{-1}C_2(X) \end{aligned}$$

where the remaining steps are standard (see e.g. [13] Chapter 9).

It seems likely that if the sequence in Theorem 14 satisfies $\mathbf{E}\,\| \Sigma_1^n \varepsilon_i x_i \| \geqq cn^{1/p}$ for some constant c, then Lemma 11 is too weak. However, we have not found a way of exploiting the growth in this case to obtain a better bound.

Appendix. The Hilbert space does not have property B

In 1961, Bishop and Phelps [3] showed that given any Banach space X, the set of norm-attaining continuous linear functionals on X is dense in X^*, where a norm-attaining functional is simply a functional $x^* \in X^*$ for which there exists $x \in X$ such that $|x^*(x)| = \| x \|_{X^*}$. They asked in their paper to what extent this result could be generalized. That is, for which pairs of Banach spaces X, Y is it true that the set of norm-attaining linear operators from X to Y is dense in the set of bounded linear operators from X to Y? A bounded operator $T: X \to Y$ is said to be norm-attaining if there exists $x \in X$ such that $\| x \| = 1$ and $\| T(x) \| = \| T \| \equiv \sup\{ \| T(x') \| : \| x' \| = 1\}$. Over the years, this question has been considered by various authors.

Let us write $B(X, Y)$ for the set of bounded linear operators from X to Y, and $NB(X, Y)$ for the set of norm-attaining ones. The question asked by Bishop and Phelps appears to be too general to allow a reasonably complete answer, so Lindenstrauss [11] made the following definitions. A Banach space X is said to have property A if for all spaces Y, $NB(X, Y)$ is dense in $B(X, Y)$, and property B if for all spaces Y, $NB(Y, X)$ is dense in $B(Y, X)$. One can now ask which Banach spaces have property A and which have property B.

Bourgain [4] showed that any space with the Radon–Nikodym property also has property A, but the results for property B are less satisfactory. The Bishop–

Phelps theorem states that **R** has property B. However, there are several examples of spaces for which it is not known whether property B holds.

In the paper mentioned above, Lindenstrauss gave a criterion for a space to have property B, and asked whether every reflexive space has it. Johnson and Wolfe [8] asked the same question, and also asked whether there is any classical space which fails property B. The result of Lindenstrauss above was used by Partington [14] to show that any Banach space can be equivalently renormed so as to have property B. He remarked that it was not known whether l_1 or l_2 have the property. Properties related to property B have also been considered by Finet and Schachermayer [5] and Finet [6].

In this appendix we show that l_2 fails property B. In fact, the proof, which is straightforward, applies equally to l_p whenever $1 < p < \infty$, so the result is presented in that form.

THEOREM. *When $1 < p < \infty$, the space l_p does not have property B.*

PROOF. Let X be the space of real sequences $(a_i)_1^\infty$ such that if $(a_i^*)_1^\infty$ is the positive decreasing rearrangement of $(a_i)_1^\infty$ (that, is, the decreasing rearrangement of $(|a_i|_1^\infty)$, then

$$\lim_{N\to\infty}\left(\sum_{i=1}^N a_i^* \Big/ \sum_{i=1}^N i^{-1}\right) = 0.$$

For $\mathbf{a} = (a_i)_1^\infty \in X$, set

$$\|\mathbf{a}\| = \max_{N\to\infty}\left(\sum_{i=1}^N a_i^* \Big/ \sum_{i=1}^N i^{-1}\right).$$

It is easily seen that $\|\cdot\|$ is a 1-symmetric norm on X.

Finally, let $T: X \to l_p$ be the formal identity map. One can prove the following easy facts.

(i) X is a Banach space;

(ii) T is a bounded linear operator;

(iii) if $S: X \to l_p$ is any norm-attaining map, then there is some $n \in \mathbf{N}$ such that $S\mathbf{e}_n = 0$.

We shall not give the details of (i). X is easily verified to be a closed subspace of the space of sequences for which $(\Sigma_{i=1}^N a_i^*/\Sigma_{i=1}^N i^{-1})$ is bounded, with the norm as above. This space is well known, and is the dual of a Lorentz sequence space.

We shall prove that T is bounded by showing that if $\mathbf{a} \in X$ and $\|\mathbf{a}\| = 1$, then $\|T\mathbf{a}\| \leq (\Sigma_1^\infty i^{-p})^{1/p}$. In fact the following inequality is obviously enough: suppose $(a_i)_1^n$ is a monotone decreasing sequence of positive numbers such that for all $m \leqq n$, $\Sigma_1^m a_i \leqq \Sigma_1^m i^{-1}$. Then $\Sigma_1^n a_i^p \leqq \Sigma_1^n i^{-p}$.

To prove this, pick a sequence $(a_i)_1^n$ satisfying these conditions such that $\Sigma_1^n a_i^p$ is maximal. If $a_i = i^{-1}$ for all i then there is nothing to prove. Otherwise pick a minimal j such that $a_j < j^{-1}$, and a maximal k such that $\Sigma_{i=1}^k a_i = \Sigma_{i=1}^k i^{-1}$. There must exist such a k and it is greater than j, for if it were not, we could increase α_j to produce a new sequence satisfying the given conditions with $\Sigma_1^n a_i$ larger than before, contradicting our maximality assumption. Now it is obvious that $a_j < a_{j-1}$ and $a_k > a_{k+1}$. Hence, for sufficiently small ε we may replace a_j by $a_j + \varepsilon$ and a_k by $a_k - \varepsilon$ to obtain a new sequence satisfying the above conditions. But this again contradicts the maximality of $\Sigma_1^n a_i^p$, since

$$\begin{aligned}&(a_j + \varepsilon)^p + (a_k - \varepsilon)^p \\ &\quad = a_j^p + a_k^p + p(a_j^{p-1} - a_k^{p-1})\varepsilon + p(p-1)/2(a_j^{p-2} + a_k^{p-2})\varepsilon^2 + o(\varepsilon^2) \\ &\quad > a_j^p + a_k^p\end{aligned}$$

when ε is sufficiently small, as $a_j \geqq a_k$.

In order to prove the third fact, which is of course the important one, we use the fact that for $1 < p < \infty$, l_p is strictly convex. Suppose then that $S : X \to l_p$ is a norm-attaining operator. Pick $\mathbf{a} = (a_i)_1^\infty \in X$ such that $\| \mathbf{a} \| = 1$ and $\| S\mathbf{a} \| = \| S \|$. Without loss of generality $(a_i)_1^\infty$ is a positive monotone decreasing sequence. Note that for every n, $a_n \leqq n^{-1} \Sigma_1^n i^{-1}$, and certainly $a_n \leqq a_1 \leqq 1$. If the sequence $(a_i)_1^\infty$ is eventually zero, then let n be the minimal index such that $a_n = 0$. Then for $\delta \leqq n^{-1}$ we have that $\| \mathbf{a} \pm \delta \mathbf{e}_n \| = \| \mathbf{a} \| \leqq 1$. Suppose on the other hand that a_n is never zero. Find m such that for any $n \geqq m$, $\Sigma_1^n a_i + 1 \leqq \Sigma_1^n i^{-1}$. There must certainly exist an $n \geqq m$ such that $a_n < a_{n-1}$, because if there were not, the expressions $\Sigma_1^N a_i / \Sigma_1^N i^{-1}$ would not even be bounded. For such an n we have that if $0 < \delta < a_{n-1} - a_n \leqq 1$, then $\| \mathbf{a} \pm \delta \mathbf{e}_n \| \leqq 1$. (We have used the fact that $a_{n-1} - a_n \leqq 1$.) Now since l_p is strictly convex, either $S\mathbf{e}_n = 0$ or

$$\| S(\mathbf{a} + \delta \mathbf{e}_n) \| + \| S(\mathbf{a} - \delta \mathbf{e}_n) \| > 2 \| S\mathbf{a} \| .$$

The second possibility is ruled out because S was supposed to attain its norm at $\mathbf{a}$.

This paper will form part of the author's PhD thesis. The author would like to thank his supervisor Dr B. Bollobás for a great deal of very valuable advice, and also C. Finet for drawing his attention to the problem solved in the appendix.

References

1. D. Amir and V. D. Milman, *Unconditional and symmetric sets in n-dimensional normed spaces*, Isr. J. Math. **37** (1980), 3–20.

2. D. Amir and V. D. Milman, *A quantitative finite dimensional Krivine theorem*, Isr. J. Math. **50** (1985), 1–12.

3. E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Am. Math. Soc. **67** (1961), 97–98.

4. J. Bourgain, *On dentability and the Bishop–Phelps property*, Isr. J. Math. **28** (1977), 265–271.

5. C. Finet and W. Schachermayer, *Equivalent norms on separable Asplund spaces*, Studia Math. **93** (1988), to appear.

6. C. Finet, *Renorming Banach spaces with many projections and smoothness properties*, to appear.

7. W. T. Gowers, *Symmetric block bases in finite-dimensional normed spaces*, Isr. J. Math. **68** (1989), 193–219.

8. J. Johnson and J. Wolfe, *Norm attaining operators*, Studia Math. **65** (1979), 7–19.

9. H. König and L. Tzafriri, *Some estimates for type and cotype constants*, Math. Ann. **256** (1981), 85–94.

10. S. Kwapien, *Isomorphic characterization of inner product spaces by orthogonal series with vector coefficients*, Studia Math. **44** (1972), 583–595.

11. J. Lindenstrauss, *On operators which attain their norm*, Isr. J. Math. **1** (1963), 139–148.

12. B. Maurey, *Construction de suites symétriques*, C.R. Acad. Sci. Paris **288** (1979), 679–681.

13. V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. Vol. 1200, Springer-Verlag, Berlin, 1986.

14. J. R. Partington, *Norm attaining operators*, Isr. J. Math. **43** (1982), 273–276.

15. R. Rado, *An inequality*, J. London Math. Soc. **27** (1952), 1–6.

16. G. Schechtman, *Levy type inequality for a class of metric spaces*, in *Martingale Theory in Harmonic Analysis and Banach Spaces*, Springer-Verlag, Berlin, 1981, pp. 211–215.

17. N. Tomczak-Jaegermann, *Computing 2-summing norms with few vectors*, Ark. Math. **17** (1979), 273–277.